Math 1512 - Exam 4 Study Guide

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Contents

Summary and Disclaimer

This is a study guide for the third exam for math 1512 at the University of New Mexico (Calculus I). The exam covers sections 5.1-5.5 of Stewart's Calculus. As such, this study guide is focused on that material. I assume that the student reading this study guide is familiar with the material previously covered in calculus 1, as well as the material from college algebra and trigonometry. If a you feel that you need to review this material, you can send me an email, or take a look at Paul's Online Math notes:

https://tutorial.math.lamar.edu/

If you are not in my class, I cannot guarantee how much these notes will help you. With that said, if your TA or instructor has shared these with you, then you will most likely get some use out of them.

Methods and Techniques

The focus of this exam is on applications of integration. In particular, we focus on finding the area between two curves, finding the volume of a solid of revolution, evaluating work done by an object, and finding the average value of a function over a given interval. This means that to really master the material, you need to have a solid understanding of basic integration techniques.

Our first method deals with finding the area between two curves. For this, it is often helpful to draw a picture of the situation, but we will lay out what it looks like on paper without the picture.

Area between two curves

Given two curves $f(x)$ and $g(x)$, where they meet at two points, say a and b, we first need to find out which is bigger in the given interval. For the sake of writing it out, suppose that on $[a, b]$ we have that $f(x) \leq g(x)$. Then the area between the two is given by

$$\int_a^b g(x) - f(x)\, dx.$$

The above gives you true area, without a sign. So, if you ever get a negative sign in your final answer, it is because you took the wrong function to be the "bigger" of the two. So, just go through your work, switching them around, and you will get the right answer at the end.

Next, we are going to work through how we find the area obtained by rotating a curve about an axis. We begin by the method of washers, which is really made to rotate curves around the x-axis. This can be described as follows:

Volume by Washers

To find the volume of a curve $f(x)$ for $a \leq x \leq b$ rotated about the x-axis, we use the formula

$$V = \int_a^b \pi(f(x))^2 \, dx$$

where V denotes the volume.

The other method to find the volume of a solid of revolution is the method of shells. This is best used when rotating about the y-axis. This gives us the following method:

Volume by Shells

To find the volume of a curve $f(x)$ for $a \leq x \leq b$ rotated about the y-axis, we use the formula

$$V = \int_a^b 2\pi x f(x) \, dx$$

where V denotes the volume.

Next up, we go through how to compute the work done by an object using calculus.

Evaluating Work

To evaluate the amount of work done to move an object between two points, say a and b, let $F(x)$ be the amount of force needed to move an object at a point x. Then the work W is given by

$$W = \int_a^b F(x) \, dx.$$

Finally, we go through how to find the average value of a function.

Finding the Average of a Function

The average value of a function $f(x)$ between two points, say a and b, is given by

$$\frac{1}{b-a} \int_a^b f(x) \, dx.$$

Worked Examples

We will now work through some examples.

Example: Find the work done by moving an object from a start position ($x = 0$) to a position 10 meters away ($x = 10$) if the force required to move the object at any given point is given by

$$F(x) = x^3 - 100x.$$

We begin by integrating directly. From here, we get that the work is equal to

$$\int_0^{10} x^3 - 100x \; dx = \frac{x^4}{4} - 50x \Big|_0^{10} = 2000.$$

So, the work done by moving the object is 2000 of whatever the appropriate units in this circumstance are.

Next, we will go through an example of finding the volume by using washers.

Example: Find the volume of the surface of revolution obtained by rotating the function $x^3 - 25x$ about the x-axis for $0 \leq x \leq 5$.
To do this, we first note that we are rotating about the x-axis, so we are using the washers method. Setting up our integral, we get that the volume is given by

$$V = \int_0^5 \pi(x^3 - 25x)^2 \; dx = \int_0^5 \pi x^6 - \pi 50x^4 + \pi 625x^2.$$

Evaluating then gives us that

$$V = \pi\frac{x^7}{7} - \pi\frac{10x^5}{5} + 625\pi\frac{x^3}{3}\Big|_0^5 = \frac{78125}{7}\pi + 6250\pi + \frac{78125}{3}\pi.$$

Practice Problems

These practice problems are separate from the unsolved problems. They should be used to make sure that you are confident with the material, and are of approximately the same level of difficulty as the unsolved questions. They also include worked solutions, unlike the unsolved questions section.

1. Find the area between $\cos(x)$ and $x = 1$ for $0 \leq x \leq 2\pi$.
2. Find the volume of the surface of revolution obtained by rotating $\cos(x^2)$ about the y-axis for $0 \leq x \leq \sqrt{\frac{\pi}{2}}$.

If it is requested, I will add more practice problems.

Practice Problem Solutions

1. Find the area between $\cos(x)$ and $x = 1$ for $0 \leq x \leq 2\pi$.

   Solution: We begin by setting the two equations equal. This gives us $\cos(x) = 1$ where $0 \leq x \leq 2\pi$. This happens at $x = 0$ and $x = 2\pi$. So, our integral will be evaluated from 0 to 2π. From here, we see that $x = 1$ is larger on this interval. So, the area is given by
   $$A = \int_0^{2\pi} 1 - \cos(x)\ dx$$
   which we can evaluate fairly easily. This gives us
   $$A = x - \sin(x)\Big|_0^{2\pi} = 2\pi.$$
   So, the area between the two curves between 0 and 2π is 2π.

2. Find the volume of the surface of revolution obtained by rotating $\cos(x^2)$ about the y-axis for $0 \leq x \leq \sqrt{\frac{\pi}{2}}$.

   Solution: Since we are rotating about the y-axis, we use the method of shells. This gives us that the volume is given by the integral
   $$V = \int_0^{\sqrt{\frac{\pi}{2}}} 2\pi x \cos(x^2)\ dx.$$
   From here, we use a u-substitution to solve this. Setting $u = x^2$, we get that $du = 2x\ dx$, so our integral is the same as
   $$V = \int_{x=0} x = \sqrt{\frac{\pi}{2}} \pi \cos(u)\ du$$
   Changing the bounds into terms of u, we get that if $x = 0$ then $u = 0$, and if $x = \sqrt{\frac{\pi}{2}}$, then $u = \frac{\pi}{2}$. So,
   $$V = \int_0^{\frac{\pi}{2}} \pi \cos(u)\ du = \sin(u)\Big|_0^{\frac{\pi}{2}} = 1.$$
   So, the volume of the given surface is 1.

Unsolved Questions

Here is a list of 20 unsolved questions which I feel are of similar difficulty to what might be asked of you on an exam.

1. Find the area between $\sin(x)$ and $\cos(x)$ where $0 \leq x \leq 2\pi$.
2. Find the area between the y-axis and $\sin(x)$ where $0 \leq x \leq \pi$
3. Find the area between $\cos^2(x)$ and $-\sin^2(x)$ where $\pi \leq x \leq 5$.
4. Find the area between x^3 and $-3x$ where $-1 \leq x \leq 0$.
5. Find the volume obtained by rotating $x^3 - 3x$ about the x-axis for $0 \leq x \leq \sqrt{3}$.
6. Find the volume obtained by rotating $\sqrt{\sin(x)}$ about the x-axis for $0 \leq x \leq \pi$.
7. Find the volume obtained by rotating $\sqrt{\cos(\pi x)}$ about the x-axis for $0 \leq x \leq \frac{1}{2}$.
8. Find the volume obtained by rotating $\sqrt{9x - x^3}$ about the x-axis for $0 \leq x \leq 3$.
9. Find the volume obtained by rotating $\sin(\pi x^2)$ about the y-axis for $0 \leq x \leq 1$.
10. Find the volume obtained by rotating $\frac{\sec(x)\tan(x)}{x}$ about the y-axis for $\frac{\pi}{6} \leq x \leq \frac{\pi}{3}$.
11. Find the volume obtained by rotating $\frac{\sin(x)}{x}$ about the y-axis for $0 \leq x \leq \pi$.
12. Find the volume obtained by rotating $9 - x^2$ about the y-axis for $0 \leq x \leq 3$.
13. If the force required to move an object is given by $F(x) = \sin(2\pi x)$ find the work done in moving the object from $x = 0$ to $x = 10$.
14. If the force required to move an object is given by $F(x) = x^5 - 3x^3 - 1$ find the work done in moving the object from $x = 0$ to $x = 20$.
15. If the force required to move an object is given by $F(x) = x^4 - \sqrt{x} + \sqrt[5]{x^2 + 1} - \sqrt[3]{x^2}$ find the work done in moving the object from $x = 0$ to $x = 5$.
16. If the force required to move an object is given by $F(x) = x^4 - 3x - \frac{1}{x^2}$ find the work done in moving the object from $x = 1$ to $x = 5$.
17. Find the average value of $f(x) = \cos(x) + \sin(x)$ between $x = 0$ and $x = \pi$.
18. Find the average value of $f(x) = x^3 - 3x$ between $x = 3$ and $x = 5$.
19. Find the average value of $f(x) = x^5 - 3x - \frac{5}{x^2} + 2\sqrt{x}$ between $x = 6$ and $x = 7$.
20. Find the average value of $f(x) = \sec^2(x)$ between $x = \frac{3\pi}{4}$ and $x = 2\pi$.